

Super Distributions, Analytic and Algebraic Super Harish-Chandra pairs

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1 Introduction

The purpose of this paper is to extend the theory of Super Harish-Chandra pairs, originally developed by Koszul for Lie supergroups [14], to analytic and algebraic supergroups, in order to obtain information also about their representations. Along the same lines, we also want to define the distribution superalgebra for algebraic and analytic supergroups and study its relation with the universal enveloping superalgebra in analogy with Kostant's treatment for the differential category [13].

Our intention is to provide different but equivalent approaches to the study of analytic and algebraic supergroups and their actions over fields of characteristic zero.

We realize that in both cases the theory is very similar to the differential one, however given some crucial differences between the smooth category and the analytic and algebraic one, we believe the present work is justified, given the importance of this theory for practical purposes together with the lack of an appropriate and complete available reference, though we are aware that a good step towards a complete clarification of these issues, for the analytic setting only, appears in the papers [21], [22].

This paper was put on the web on June 2011. Since then Masuoka has published a more general result and in his paper [15] has quoted our work.

Since our methods are somewhat different from Masuoka's we hope our work deserves a place in the literature.

This paper is organized as follows.

In Section 2 we describe the superalgebra of distributions of an analytic or an algebraic supergroup, establishing its relation with the universal enveloping superalgebra.

In Section 3 we establish the equivalence between the category of analytic or algebraic super Harish-Chandra pairs and the category of analytic or affine algebraic supergroups under suitable hypothesis for the ground field.

In Section 4 we provide an equivalent approach to the study of the actions of supergroups, via the super Harish-Chandra pairs (SHCP).

For all the definitions and main results in supergeometry expressed with our notation, we refer the reader to [8] ch. 2 or [2] ch. 1, 4, 10. In particular we shall employ both the sheaf theoretic and the functor of points approach to supergeometry. On this we invite the reader to consult the classical references [3], [15], [20].

Acknowledgements. We wish to thank prof. Varadarajan, for suggesting the problem and prof. Cassinelli and prof. Gavarini for helpful discussions.

2 The superalgebra of distributions

We start giving the definition of distribution and distribution superalgebra. Our treatment is general enough to accommodate the two very different categories of supermanifolds and superschemes. For the classical definitions we send the reader to [11] pg 95, [5] II §4, no. 6 and [6]. For the basic definitions of supergeometry we refer the reader to [15], [20], [3], [8].

2.1 Distributions

Let k be the ground field.

Let $X = (|X|, \mathcal{O}_X)$ be an analytic supermanifold or an algebraic superscheme over the field k .¹

¹If X is an analytic supermanifold, $k = \mathbb{R}$ or $k = \mathbb{C}$ or even $k = \mathbb{Q}_p$, the p -adic numbers (see for example [18]). If X is a superscheme, k is a generic field.

Let $X(k)$ be the k -points of X , that is $X(k) = \text{Hom}(k^{0|0}, X)$ in the functor of points notation. For an analytic supermanifold X we have that its k -points $X(k)$ are identified with the topological points $|X|$, while for X a superscheme the k -points, are in one to one correspondence with the rational points, that is, the points $x \in |X|$ for which $\mathcal{O}_{X,x}/m_{X,x} \cong k$, $m_{X,x}$ being the maximal ideal in the stalk $\mathcal{O}_{X,x}$.

Definition 2.1. A *distribution supported at $x \in X(k)$ of order at most n* is a morphism $\phi : \mathcal{O}_{X,x} \longrightarrow k$, with $m_{X,x}^{n+1} \subset \ker(\phi)$ for some n . The set of all distributions at x of order n is denoted as $D_n(X, x)$, while $D(X, x)$ denotes all distributions supported at x . Both $D_n(X, x)$ and $D(X, x)$ have a natural super vector space structure.

We also define:

$$D(X) = \bigcup_{x \in X(k)} D(X, x).$$

as the *distributions of finite order* of X . Also $D(X)$ has a natural super vector space structure.

Observation 2.2. 1. Notice that most immediately:

$$D_n(X, x) \cong (\mathcal{O}_{X,x}/m_{X,x}^{n+1})^*$$

since if $\phi \in D_n(X, x)$, $\phi(m_{X,x}^{n+1}) = 0$, hence it factors and becomes an element in $(\mathcal{O}_{X,x}/m_{X,x}^{n+1})^*$. Notice furtherly that:

$$D_0(X, x) = k, \quad D_1(X, x) = k \oplus (m_{X,x}/m_{X,x}^2)^*.$$

Hence $D_1(X, x)^+ := (m_{X,x}/m_{X,x}^2)^*$ becomes identified with the tangent space to X at the point x .

2. If X is an affine algebraic superscheme, $\mathcal{O}(X)$ the superalgebra of the global sections of its structural sheaf, a distribution supported at x of order n can be equivalently seen as a morphism $\phi : \mathcal{O}(X) \longrightarrow k$, with $m_x^n \subset \ker(\phi)$, where $m_x := \{\phi \in \mathcal{O}(X) \mid \phi(x) = 0\}$ is the maximal ideal of all the functions vanishing at x , where as usual in supergeometry $f(x)$ simply means the image in $\mathcal{O}_{X,x}/m_{X,x}$ of the element $f \in \mathcal{O}(X)$ under the natural morphisms: $\mathcal{O}(X) \longrightarrow \mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{X,x}/m_{X,x} \cong k$. (Notice that since x is rational, we have $\mathcal{O}(X) = k \oplus m_x$ and $\mathcal{O}_{X,x}/m_{X,x} \cong k$).

We leave to the reader to check that the two given definitions of distributions are essentially the same in this case.

3. If X is a smooth supermanifold, that is, if we are in the differential category, we can view a point supported distribution as a morphism $\phi : \mathcal{O}(X) \rightarrow \mathbb{R}$, $J_x^n \subset \ker(\phi)$, where J_x is the maximal ideal corresponding to the point $x \in |X|$ (see [13] and [2] 4.7), thus recovering the same definition as in (2) for the affine algebraic category. This is one of the many analogies between the category of affine supervarieties and smooth supermanifolds.

Example 2.3. *The distributions on $k^{p|q}$. ($\text{char}(k) = 0$). Consider the superspace $k^{p|q}$ (both in the analytic and affine algebraic context). Let $x_1 \dots x_p, \xi_1 \dots \xi_q$ denote the global coordinates and $J_0 = (x_1 \dots x_p, \xi_1 \dots \xi_q)$ the maximal ideal in the stalk $\mathcal{O}_{X,0}$ at the origin. We have that*

$$\mathcal{O}_{X,0}/J_0^{n+1} \cong \text{span}_k \{1, x_1^{i_1} \dots x_p^{i_p} \xi_1^{i_{p+1}} \dots \xi_q^{i_{p+q}}, \sum i_k = n\}.$$

Let X^I denote the monomial $x_1^{i_1} \dots x_p^{i_p} \xi_1^{i_{p+1}} \dots \xi_q^{i_{p+q}}$, $I = (i_1 \dots i_{p+q})$. Since the distributions at 0 of order n are the dual of the super vector space $\mathcal{O}_{X,0}/J_0^{n+1}$, we have that a basis for the super vector space of distributions at the point 0 is given by ϕ_J such that $\phi_J(X_I) = \delta_{IJ}$, with $I = (i_1 \dots i_{p+q})$, $J = (j_1 \dots j_{p+q})$ multiindices, $\sum i_k = \sum j_k = n$. So we have:

$$\phi_{j_1 \dots j_{p+q}}(f) = \frac{1}{j_1! \dots j_{p+q}!} \left(\frac{\partial}{\partial x_1}\right)^{j_1} \dots \left(\frac{\partial}{\partial x_p}\right)^{j_p} \left(\frac{\partial}{\partial \xi_q}\right)^{j_{p+1}} \dots \left(\frac{\partial}{\partial \xi_1}\right)^{j_{p+q}}(f)(0).$$

2.2 The superalgebra of distributions of an analytic supermanifold

In this section we want to characterize the distributions for an analytic supermanifold $M = (|M|, \mathcal{O}_M)$ in the following way. Distributions at the point $x \in |M|$ are the elements in $\mathcal{O}_{M,x}^*$, whose kernel contains an ideal of finite codimension in analogy with Kostant's treatment for the smooth category (see [13]). Let us see this more in detail, we start with a lemma.

Lemma 2.4. *Let $M = (|M|, \mathcal{O}_M)$ be an analytic supermanifold, $x \in |M|$, I_x the ideal in $\mathcal{O}_{M,x}$ of the sections vanishing at x . For each positive integer p , I_x^p is an ideal of finite codimension.*

Proof. It follows from the Taylor expansion formula. In fact, every element f in $\mathcal{O}_{M,x}$ can be written as $f = \sum_I f_I \theta^I$, where f_I is an element in the classical

stalk of germs of holomorphic functions $\mathcal{H}_{M,x}$. For each positive integer q , a germ f_I can in turn be written as

$$f_I(z) = f_I(x) + \sum_{K, |K|=1}^{q-1} (\partial_K f_I)(x) z^K + \sum_{J, |J|=q} z^J h_{I,J}(z)$$

where I, J, K are multiindices. Hence we can write

$$f = \sum_I \left(f_I(x) + \sum_{R, |R+I| < p} (\partial_R f_I)(x) z^R \right) \theta^I + \sum_{|I+R|=p} h_{I,R}(z) z^R \theta^I.$$

From this formula, it follows that the elements in I_x^p are generated by the monomials $\{z^K \theta^I\}_{|K+I| \leq p}$. \square

Proposition 2.5. *An ideal J in $\mathcal{O}_{M,x}$ has finite codimension if and only if there exists an integer $p > 0$ such that $I_x^p \subseteq J$.*

Proof. The “if” part follows from the previous lemma. For the “only if” part we reason as follows. Consider the descending chain of ideals $J + I_x^p \supseteq J + I_x^{p+1}$. Since J has finite codimension there exists q such that $J + I_x^q = J + I_x^{q+1}$. From this it follows that $I_x^q \subseteq J + I_x^q \cdot I_x$. Since, by the previous lemma, I_x^q is finitely generated we can apply the super version of Nakayama lemma (see [20]) and we get $I_x^q \subseteq J$. \square

We have then obtained the following result, which establishes a parallelism with the smooth category.

Theorem 2.6. *The distributions on an analytic supermanifold M supported at a point x correspond to the morphisms $f : \mathcal{O}_{M,x} \longrightarrow k$, whose kernel contains an ideal of finite codimension.*

2.3 The distributions of a supergroup at the identity

We now want to restrict our attention to the distributions of a supergroup (analytic or algebraic) at the identity element $e \in G(k)$.

As a consequence of the observation 2.2 we have that:

$$D_1(G, e)^+ \cong (m_{G,e}/m_{G,e}^2)^* \cong T_e(G) = \text{Lie}(G).$$

It is only natural to expect $D(G, e)$ to be identified with $\mathcal{U}(\mathfrak{g})$, with $\mathfrak{g} = \text{Lie}(G)$. This is true, as we shall see, provided we exert some care.

As we remarked in the definition 2.1 the distributions at the identity are a super vector space, however there is a natural additional superalgebra structure that we can associate to the super vector space of distributions, by defining the *convolution product*.

Definition 2.7. Let $\phi, \psi \in D(G, e)$. We define their *convolution product* as the following morphism:

$$(\phi \star \psi)(f) = (\phi \otimes \psi)\mu^*(f), \quad f \in \mathcal{O}_{G,e}$$

where μ denotes the multiplication in the supergroup G and μ^* the corresponding sheaf morphism.

The following proposition is a straightforward check.

Proposition 2.8. *The convolution product makes $D(G, e)$ a superalgebra, its unit being the evaluation at e , $ev_e : \mathcal{O}_{G,e} \longrightarrow k$.*

We now want to examine the relation of $D(G, e)$ with the universal enveloping superalgebra (uesa) of the supergroup G . Since $D(G, e) \supset D_1(G, e)^+ \cong \text{Lie}(G)$, by the universal property of the uesa $\mathcal{U}(\mathfrak{g})$ we have a superalgebra morphism $\alpha : \mathcal{U}(\mathfrak{g}) \longrightarrow D(G, e)$.

Observation 2.9. When G is an algebraic supergroup and the characteristic of k is positive, as it happens in the classical setting, $D(G, e)$ contains more than the elements coming from $\mathcal{U}(\mathfrak{g})$ (refer to example 2.3). This is because the divided powers $X^m/m!$ are in $D(G, e)$ but not in $\mathcal{U}(\mathfrak{g})$. Again similarly, as in the classical situation, we have that any morphism $\mathcal{U}(\mathfrak{g}) \longrightarrow D(G, e)$ factors via *the universal enveloping restricted algebra* $\mathcal{U}^r(\mathfrak{g})$:

$$\mathcal{U}(\mathfrak{g}) \longrightarrow \mathcal{U}^r(\mathfrak{g}) = \mathcal{U}(\mathfrak{g})/(X^p - X^{[p]}) \longrightarrow D(G, e).$$

where $X^{[p]}$ denotes the derivation in \mathfrak{g} corresponding to p -times the derivation X (which is a derivation here, since we are in characteristic p).

Let $\text{char}(k) = 0$.

Proposition 2.10. *The morphism $\alpha : \mathcal{U}(\mathfrak{g}) \longrightarrow D(G, e)$ is an isomorphism.*

Proof. This is done essentially in the same way as in the classical setting, which is detailed in [20] ch. I for the analytic category and [5] II, 6, 1.1 for the algebraic category. \square

Proposition 2.11. *There is an isomorphism of the superalgebra of distributions on a supergroup G and the superalgebra of the left invariant differential operators on G . $\mathcal{U}(\mathfrak{g})$ is then isomorphic to the superalgebra of the left invariant differential operators on G .*

Proof. Again this proof is the same as in [20] ch. I and [5] II, 6, 1.1, for the classical setting. \square

2.4 The distributions of an affine algebraic supergroup

We now want to restrict ourselves to the case of affine algebraic supergroups. As we shall see, this algebraic setting shares many similarities with the differential one.

Consider the module of distributions $D(G)$ (see observation 2.2):

$$D(G) = \cup_{x \in G(k)} D(G, x) \subset \mathcal{O}(G)^*.$$

Definition 2.12. If $\phi = \sum \phi_{p_i}$ is a distribution with $\phi_{p_i} \in D(G, p_i)$ we say that ϕ is *supported* at $\{p_i\}$. On the whole $D(G)$ we have a well defined associative product, called the *convolution product*:

$$(\phi_p \star \phi_q)(f) = (\phi_p \otimes \phi_q) \mu^*(f)$$

and its unit is ev_e , the evaluation at the unit element: $ev_e(f) = f(e)$. μ^* denotes (as before) the comultiplication in the Hopf superalgebra $\mathcal{O}(G)$.

Observation 2.13. If ϕ_p and ϕ_q are two distributions supported at p and q respectively, then $\phi_p \star \phi_q$ is supported at pq . This is a consequence of the fact:

$$\mu^*(m_{pq}) \subset m_p \otimes \mathcal{O}(G) + \mathcal{O}(G) \otimes m_q$$

where m_x is as usual the maximal ideal of the sections in $\mathcal{O}(G)$ vanishing at $x \in G(k)$. $m_x = m_{x,0} + J_{\mathcal{O}(G)}$, that is, m_x is the sum of $m_{x,0}$ the ordinary maximal ideal corresponding to the topological rational point $x \in G(k)$ and the ideal $J_{\mathcal{O}(G)}$ generated by the odd sections in $\mathcal{O}(G)$.

Lemma 2.14. *Let $\phi_g \in D(G, g)$. Then there exists a unique $\phi_e \in D(G, e)$ such that $\phi_e = ev_{g^{-1}} \star \phi_g$.*

Proof. Since $\phi = (ev_g \star ev_{g^{-1}}) \star \phi$, define $\phi_e = ev_{g^{-1}} \star \phi \in D(G, e)$. \square

Proposition 2.15. *$D(G)$ is super Hopf algebra with comultiplication Δ , counit η and antipode S given by:*

$$\Delta(\phi_g)(f \otimes g) := \phi_g(f \cdot g) \quad \eta(\phi_g)(f) := \phi_g(ev_e(f))$$

$$S(\phi_g)(f) := \phi_g(i^*(f)),$$

where $i : G \longrightarrow G$ denotes the inverse morphism.

Proof. Direct check. \square

Let $k|G|$ be the group algebra corresponding to the ordinary group $G(k)$. In other words

$$k|G| = \left\{ \sum_{g \in G(k), \lambda_g \in k} \lambda_g g \right\}.$$

Proposition 2.16. *We have a linear isomorphism:*

$$\begin{array}{ccc} \Psi : D(G) & \longrightarrow & k|G| \otimes \mathcal{U}(g) \\ \phi_g & \mapsto & g \otimes \phi_e \end{array}$$

that endows $k|G| \otimes \mathcal{U}(\mathfrak{g})$ of a Hopf superalgebra structure. This structure is induced by the natural Hopf structures on the group algebra $k|G|$ and $\mathcal{U}(\mathfrak{g})$:

$$\Delta_{k|G|}(g) = g \otimes g, \quad \Delta_{\mathcal{U}(\mathfrak{g})}(U) = U \otimes 1 + 1 \otimes U, \quad g \in G(k), U \in \mathfrak{g}.$$

The superalgebra structure is defined as:

$$(g \otimes X)(h \otimes Y) = gh \otimes (h^{-1}X)Y, \quad g \in G(k), \quad X, Y \in \mathcal{U}(\mathfrak{g})$$

with $h^{-1}X := ev_{h^{-1}} \star X \star ev_h$, (by 2.10 we identify distributions at e with elements in $\mathcal{U}(\mathfrak{g})$).

Proof. This is done with a direct check. We just point out that it is enough to do such check just on generators. \square

3 Super Harish-Chandra Pairs

The theory of Super Harish-Chandra Pairs (SHCP) that we shall develop presently provides an equivalent way to approach the analytic or affine algebraic supergroups.

3.1 Definition of SHCP

Any time we say *supergroup* we mean an analytic or an affine algebraic supergroup over a field k of characteristic zero.

Definition 3.1. Suppose (G_0, \mathfrak{g}) are respectively a group (analytic or affine algebraic) and a super Lie algebra. Assume that:

1. $\mathfrak{g}_0 \simeq \text{Lie}(G_0)$,
2. G_0 acts on \mathfrak{g} and this action restricted to \mathfrak{g}_0 is the adjoint representation of G_0 on $\text{Lie}(G_0)$. Moreover the differential of such action is the Lie bracket. We shall denote such an action with Ad or as $g.X$, $g \in G_0$, $X \in \mathfrak{g}$.

Then (G_0, \mathfrak{g}) is called a *super Harish-Chandra pair (SHCP)*.

A *morphism* of SHCP is simply a pair of morphisms $\psi = (\psi_0, \rho^\psi)$ preserving the SHCP structure that is:

1. $\psi_0 : G_0 \rightarrow H_0$ is a group morphism (in the analytic or algebraic category);
2. $\rho^\psi : \mathfrak{g} \rightarrow \mathfrak{h}$ is a super Lie algebra morphism
3. ψ_0 and ρ^ψ are compatible in the sense that:

$$\rho^\psi|_{\mathfrak{g}_0} = d\psi_0 \quad \text{Ad}(\psi_0(g)) \circ \rho^\psi = \rho^\psi \circ \text{Ad}(g)$$

When G_0 is an analytic group we shall speak of an *analytic SHCP*, when G_0 is an affine algebraic group of an *algebraic SHCP*.

We would like to show that the category of (analytic or algebraic) SHCP (denoted with (shcps)) is equivalent to the category of supergroups (analytic or algebraic), denoted with (sgrps). In order to do this we start by associating in a natural way a supergroup to a SHCP.

Definition 3.2. Let (G_0, \mathfrak{g}) be a SHCP. The sheaf \mathcal{O}_{G_0} of the ordinary group G_0 carries a natural action of $\mathcal{U}(\mathfrak{g}_0)$, since the elements of $\mathcal{U}(\mathfrak{g}_0)$ act on the sections in $\mathcal{O}_{G_0}(U)$ as left invariant differential operators. We define $\mathcal{O}_G(U)$ as:

$$\mathcal{O}_G(U) := \text{Hom}_{\mathcal{U}(\mathfrak{g}_0)}(\mathcal{U}(\mathfrak{g}), \mathcal{O}_{G_0}(U)), \quad U \subset_{\text{open}} G_0.$$

Proposition 3.3. *The assignment $U \mapsto \mathcal{O}_G(U)$ is a sheaf of superalgebras on G_0 , where the superalgebra structure on $\mathcal{O}_G(U)$ is given by:*

$$f_1 \cdot f_2 = m_{\mathcal{O}_{G_0}} \circ (f_1 \otimes f_2) \circ \Delta_{\mathcal{U}(\mathfrak{g})}$$

and the restriction morphisms $\rho_{UV} : \mathcal{O}_G(U) \longrightarrow \mathcal{O}_G(V)$ are $\rho_{UV}(f) := \tilde{\rho}_{UV} \circ f$ where $\tilde{\rho}_{UV}$ are the restrictions of the ordinary sheaf \mathcal{O}_{G_0} .

Proof. The check $f_1 \cdot f_2$ is an associative product is routine, while the sheaf property comes from the fact \mathcal{O}_{G_0} is an ordinary sheaf. \square

We now show that (G_0, \mathcal{O}_G) is a superspace, by showing that is *globally split*, in other words that:

$$\mathcal{O}_G(U) \cong \mathcal{O}_{G_0}(U) \otimes \wedge(\mathfrak{g}_1).$$

Theorem 3.4. 1. *The map*

$$\begin{aligned} \hat{\gamma} : \mathcal{U}(\mathfrak{g}_0) \otimes \wedge(\mathfrak{g}_1) &\rightarrow \mathcal{U}(\mathfrak{g}) \\ X \otimes Y &\mapsto X \cdot \gamma(Y) \end{aligned}$$

is an isomorphism of super left $\mathcal{U}(\mathfrak{g}_0)$ -modules, where

$$\begin{aligned} \gamma : \wedge(\mathfrak{g}_1) &\rightarrow \mathcal{U}(\mathfrak{g}) \\ X_1 \wedge \cdots \wedge X_p &\mapsto \frac{1}{p!} \sum_{\tau \in S_p} (-1)^{|\tau|} X_{\tau(1)} \cdots X_{\tau(p)} \end{aligned}$$

is the symmetrizer map, $|\tau|$ denotes the parity of the permutation τ .

2. (G_0, \mathcal{O}_G) is globally split i. e. for each open subset $U \subseteq G_0$, there is an isomorphism of superalgebras

$$\mathcal{O}_G(U) \simeq \text{Hom}(\wedge(\mathfrak{g}_1), \mathcal{O}_{G_0}(U)) \simeq \mathcal{O}_{G_0}(U) \otimes \wedge(\mathfrak{g}_1)^*. \quad (1)$$

Hence \mathcal{O}_G carries a natural \mathbb{Z} -gradation.

Proof. (1) is an application of Poincaré-Birkhoff-Witt (PBW) theorem (see [20]), while for (2) consider the following map:

$$\begin{aligned}\phi_U : \mathcal{O}_G(U) &\rightarrow \text{Hom}(\wedge(\mathfrak{g}_1), \mathcal{O}_{G_0}(U)) \\ f &\rightarrow f \circ \gamma\end{aligned}$$

Since γ is a supercoalgebra morphism, ϕ_U is a superalgebra morphism. In fact:

$$\phi_U(f_1 \cdot f_2) = m \circ f_1 \otimes f_2 \circ \Delta_{\mathcal{U}(\mathfrak{g})} \circ \gamma = m \circ f_1 \otimes f_2 \circ (\gamma \otimes \gamma) \Delta_{\mathcal{U}(\mathfrak{g})} = \phi_U(f_1) \phi_U(f_2).$$

The fact that ϕ_U is a superalgebra isomorphism follows at once from $\mathcal{U}(\mathfrak{g}_0)$ -linearity. \square

As an almost immediate consequence of the previous theorem we have the following corollary.

Corollary 3.5. *If G_0 is an analytic manifold (resp. algebraic scheme), then (G_0, \mathcal{O}_G) is a superspace.*

In the next sections we will complete the task of showing (G_0, \mathcal{O}_G) is a supergroup, by providing explicit expression for the multiplication, unit and inverse. This will lead to the main result of the paper, namely the equivalence of categories between the SHCP and supergroups. We now state the main result of the paper and then we shall prove it with different methods in the next sections, since at this points the analytic and algebraic categories diverge and require a dramatically different treatment.

Theorem 3.6. *Let k be a field of characteristic zero, algebraically closed if we are in the algebraic category. Define the functors*

$$\begin{aligned}\mathcal{H} : (\text{sgrps}) &\rightarrow (\text{shcps}) \\ G &\rightarrow (G_0, \text{Lie}(G)) \\ \phi &\rightarrow (|\phi|, (d\phi)_e) \\ \\ \mathcal{K} : (\text{shcps}) &\rightarrow (\text{sgrps}) \\ (G_0, \mathfrak{g}) &\rightarrow \bar{G} := (G_0, \text{Hom}_{\mathcal{U}(\mathfrak{g}_0)}(\mathcal{U}(\mathfrak{g}), \mathcal{O}_{G_0})) \\ \psi = (\psi_0, \rho^\psi) &\rightarrow f \mapsto \psi_0^* \circ f \circ \rho_\psi\end{aligned}$$

where G and (G_0, \mathfrak{g}) are objects and ϕ, ψ are morphisms of the corresponding categories (in the definition of \mathcal{H} , G_0 is the ordinary group underlying G). Then \mathcal{H} and \mathcal{K} define an equivalence between the categories of supergroups (analytic or algebraic) and super Harish-Chandra pairs (analytic or algebraic).

3.2 Analytic SHCP

Let $k = \mathbb{R}$ or \mathbb{C} .

For analytic SHCP it is relatively easy to define a supergroup structure on the superspace (G_0, \mathcal{O}_G) we have defined above, by mimicking what happens in the smooth case. In fact for an analytic ordinary group G_0 , the action of $\mathcal{U}(\mathfrak{g}_0)$ on \mathcal{O}_{G_0} is given by:

$$(\tilde{D}_Z \cdot f)(g) = f(ge^{tZ}), \quad Z \in \mathfrak{g}_0, \quad f \in \mathcal{O}_{G_0}(U)$$

where e^{tZ} denotes the one-parameter subgroup corresponding to the element $Z \in \mathfrak{g}_0$.

Proposition 3.7. *(G_0, \mathcal{O}_G) is an analytic supergroup where the multiplication μ , inverse i and unit e and are defined via the corresponding sheaf morphisms as follows.*

$$[\mu^*(f)(X, Y)](g, h) = [f((h^{-1}.X)Y)](gh) \quad (2)$$

$$[i^*(f)(X)](g^{-1}) = [f(g^{-1}.\overline{X})](g) \quad (3)$$

$$e^*(f) = [f(1)](e) \quad (4)$$

for $f \in \mathcal{O}_G(U)$, $g, h \in |G|$, where $|G|$ is the topological space underlying G_0 . \overline{X} denotes the antipode in $\mathcal{U}(\mathfrak{g})$.

Note. We shall discuss the peculiar form of μ^* , i^* , e^* in remark 3.14.

Proof. The proof of this result is the same as in the differential smooth setting, where everything is defined in the same way (see [2] ch. 7). In particular to prove that μ^* , i^* , e^* are $\mathcal{U}(\mathfrak{g}_0)$ -morphisms is harder than the verification of the compatibility conditions and the Hopf superalgebra properties. As an example, let us verify μ is well defined the other properties being essentially the same type of calculation. Due to PBW theorem, it is enough to prove \mathfrak{g}_0 -linearity. Let hence $Z \in \mathfrak{g}_0$

$$\begin{aligned} \mu^*(f)(ZX, Y)(g, h) &= f(h^{-1}(ZX)Y)(gh) \\ &= f((h^{-1}.Z)(h^{-1}.X)Y)(gh) \\ &= \tilde{D}_{h^{-1}.Z} [f((h^{-1}.X)Y)](gh) \end{aligned}$$

on other hand

$$\begin{aligned}
\left[(\tilde{D}_Z \otimes \text{id}) (\mu^* (f) (X, Y)) \right] (g, h) &= \frac{d}{dt} \Big|_{t=0} f ((h^{-1} X) Y) (g e^{tZ} h) \\
&= \frac{d}{dt} \Big|_{t=0} f ((h^{-1} X) Y) (g h e^{t(h^{-1} Z)}) \\
&= \tilde{D}_{h^{-1} Z} [f ((h^{-1} X) Y)] (gh).
\end{aligned}$$

Similarly for the left entry, one finds:

$$\begin{aligned}
\mu^* (f) (X, ZY) (g, h) &= f ((h^{-1} X) ZY) (gh) \\
&= f (Z(h^{-1} X) Y + [h^{-1} X, Z] Y) (gh) \\
&= \tilde{D}_Z (f ((h^{-1} X) Y)) (gh) + \\
&\quad f ([h^{-1} X, Z] Y) (gh)
\end{aligned}$$

and

$$\begin{aligned}
\frac{d}{dt} \Big|_{t=0} \mu^* (f) (X, Y) (g, h e^{tZ}) &= \frac{d}{dt} \Big|_{t=0} f (((h e^{tZ})^{-1} X) Y) (g h e^{tZ}) \\
&= \left[\tilde{D}_Z f ((h^{-1} X) Y) \right] (gh) + \\
&\quad + f ([h^{-1} X, Z] Y) (gh).
\end{aligned}$$

□

We are now ready for the proof of theorem 3.6 in the analytic setting.

Theorem 3.8. *There is an equivalence of categories between analytic SHCP and analytic supergroups expressed by the functors \mathcal{K} and \mathcal{H} in 3.6.*

Proof. Let us first show the correspondence between morphisms. If ϕ is a morphism of analytic supergroups, it is immediate that $(|\phi|, (d\phi)_e)$ is a morphism of SHCP. Vice versa, if $\psi = (\psi_0, \rho_\psi)$ is a morphism of SHCP $(G_0, \mathfrak{g}), (H_0, \mathfrak{h})$, then $\psi^* : \mathcal{O}_H(U) \longrightarrow \mathcal{O}_G(\psi_0^{-1}(U))$ defined as $\psi^*(f) = \psi_0^* \circ f \circ \rho_\psi$ is a sheaf morphism and (ψ_0, ψ^*) is a morphism of the supergroups G and H . As one can check the assignments detailed in 3.6 establish a one-to-one correspondence between the sets of morphisms of SHCP and analytic supergroups.

We now turn to the correspondence between the objects. Let G be a supergroup and \overline{G} the supergroup obtained from the SHCP $(G_0, \text{Lie}(G))$,

where G_0 is the ordinary analytic group underlying G . As for the smooth setting, let us define the morphism $\eta: \overline{G} \rightarrow G$

$$\begin{aligned}\eta^*: \mathcal{O}_G(U) &\rightarrow \mathcal{O}_{\overline{G}}(U) = \text{Hom}_{\mathcal{U}(\mathfrak{g}_0)}(\mathcal{U}(\mathfrak{g}), \mathcal{O}_{G_0}(U)) \\ s &\mapsto \left(\overline{s}: X \rightarrow (-1)^{|X|} |(D_X s)| \right).\end{aligned}$$

Here D_X denotes the left invariant differential operator on G associated with $X \in \mathcal{U}(\mathfrak{g})$, that is $D_X = (1 \otimes X)\mu^*$. The definition is well posed as one can directly check, moreover η is a SLG morphism, i. e.

$$\eta \circ \mu_{\overline{G}} = \mu_G \circ (\eta \times \eta)$$

Indeed, for each $s \in \mathcal{O}(G)$, $X, Y \in \mathcal{U}(\mathfrak{g})$, and $g, h \in G_0$,

$$\begin{aligned}[(\eta^* \otimes \eta^*)\mu_G^*(s)](X, Y)(g, h) &= (-1)^{|X|+|Y|} |(D_X \otimes D_Y)\mu_G^*(s)|(g, h) \\ &= (-1)^{|X|+|Y|} |D_{h^{-1}.X} D_Y s|(gh) \\ &= [\eta^*(s)((h^{-1}.X)Y)](gh) \\ &= [(\mu_{\overline{G}}^* \eta^*(s))(X, Y)](g, h).\end{aligned}$$

Now the last thing to check is that η is an isomorphism. This is due to the fact that $|\eta|$ is clearly bijective and, for each $g \in G_0$, the differential $(d\eta)_g$ is bijective as

$$\begin{aligned}[(d\eta)_g(\overline{D}_{X_g})](s) &= \overline{D}_{X_g} \eta^*(s) = ev_g(\overline{D}_X \eta^*(s)) = [\overline{D}_X \eta^*(s)](1)(g) = \\ &= (-1)^{|X|} \eta^*(s)(X)(g) = |(D_X s)|(g) = D_{X_g}(s)\end{aligned}$$

where we denote with \overline{D}_X a left invariant differential operator on \overline{G} corresponding to $X \in \mathcal{U}(\mathfrak{g})$ while D_X denotes a left invariant differential operator on G .

We conclude using the inverse function theorem, which holds also for analytic supermanifolds and again this is an important difference with the algebraic setting, where we do not have this tool available. \square

Remark 3.9. *p-adic SHCP.*

p -adic supermanifolds, supergroups and SHCP can be defined through the obvious same definitions within the framework described classically by Serre in [18]. In fact since the category of p -adic manifolds resembles very closely the category of analytic manifolds, it is then only reasonable to expect that one can develop along the same lines the theory of p -adic supermanifolds. Once the basic results, like the inverse function theorem, are established, the equivalence of categories between p -adic supergroups and the p -adic SHCP will then follow through the same proof we have detailed for the analytic category.

3.3 Algebraic SHCP

We now prove our main result, namely the theorem 3.6 in the case of G an affine algebraic supergroup over a field of characteristic zero, algebraically closed.

The category of affine algebraic supergroups is equivalent to the category of commutative Hopf superalgebras, hence we need to show that there is a unique commutative Hopf superalgebra $\mathcal{O}(G)$ associated to a SHCP (G_0, \mathfrak{g}) , namely the superalgebra of the global sections of the sheaf \mathcal{O}_G as it is defined in 3.1.

We would like to state and prove the algebraic analogue of proposition 3.7. In the proof of such proposition there is an essential use of the exponential, hence we now want to formally introduce this notation in the algebraic setting, so that we can reproduce all the arguments, though being well aware that the exponential notation has a very different interpretation in the two categories analytic and algebraic.

Notice that since the exponential appears for the action of $\mathcal{U}(\mathfrak{g}_0)$ on $\mathcal{O}(G_0)$ (see beginning of sec. 3.2), the question is entirely classical and it is treated in detail in [5] ch. 2 for the algebraic setting. We shall briefly review few key facts, sending the reader to [5] for all the details.

Let G_0 be an algebraic group and A a commutative algebra, $p : A(t) \longrightarrow A$, $t^2 = 0$ the natural projection, t even. By definition $\text{Lie}(G_0)(A) = \ker G_0(p)$. Since G_0 is affine we have $G_0 \subset \text{GL}(V)$ for a suitable vector

space V , hence we can write:

$$\begin{aligned}\mathrm{Lie}(G_0)(A) &= \{1 + tZ\} \subset G_0(A(t)) \subset \mathrm{GL}(V)(A(t)) \\ &= \mathrm{GL}(V)(A) + t\mathrm{End}(V)(A)\end{aligned}$$

for suitable $Z \in \mathrm{End}(V)(A)$, where $\mathrm{End}(V)$ is the functor of points of the superscheme of the endomorphisms of the vector space V . Very often $\mathrm{Lie}(G_0)$ is identified with the subspace in $\mathrm{End}(V)$ consisting of the elements Z . As a notation device we define:

$$e^{tZ} = 1 + tZ \in G_0(A(t)).$$

Let $g \in G_0(A) = \mathrm{Hom}(\mathcal{O}(G_0), A)$, that is, g is an A -point of G_0 , and let $f \in \mathcal{O}(G_0)$. As another common notational device, we denote $g(f)$ with $f(g)$. Since A embeds naturally in $A(t)$ we can view g also as an $A(t)$ -point of G_0 and consider $f(ge^{tZ})$. We then define:

$$\frac{d}{dt}\bigg|_{t=0} f(ge^{tZ}) = b, \quad \text{where} \quad f(ge^{tZ}) = (ge^{tZ})(f) = a + bt \in A(t).$$

With such definition one sees that $\frac{d}{dt}\big|_{t=0} f(ge^{tZ})$ corresponds to the natural action of $Z \in \mathrm{Lie}(G_0)$ on $\mathcal{O}(G_0)$ via left invariant operators, that is

$$\frac{d}{dt}\bigg|_{t=0} f(ge^{tZ}) = (1 \otimes Z)\mu^*(f)$$

that we denoted with $\tilde{D}_Z f$ in the analytic category.

We now go back to the super setting and prove the analogue of proposition 3.7.

Proposition 3.10. *The superalgebra $\mathcal{O}(G) = \mathrm{Hom}(\mathcal{U}(\mathfrak{g}), \mathcal{O}(G_0))$ associated to the algebraic SHCP (G_0, \mathfrak{g}) is an Hopf superalgebra where the comultiplication μ^* , antipode i^* and counit e^* ² and are defined as follows:*

$$[\mu^*(f)(X, Y)](g, h) = [f((h^{-1} \cdot X)Y)](gh) \quad (5)$$

$$[i^*(f)(X)](g^{-1}) = [f(g^{-1} \cdot \overline{X})](g) \quad (6)$$

$$e^*(f) = [f(1)](e) \quad (7)$$

for $f \in \mathcal{O}(G)$, $g, h \in |G|$. \overline{X} denotes the antipode in $\mathcal{U}(\mathfrak{g})$.

²In analogy with proposition 3.7 we have kept the terminology μ^* , i^* , e^* , though we are not making (yet) any claim on the sheaf morphisms.

Proof. It is the same as proposition 3.7. Though the context is different, once the exponential terminology assumes a meaning for the algebraic category, the calculations are the same. \square

Next proposition shows a very natural fact, namely that given a SHCP (G_0, \mathcal{O}_G) the sheaf \mathcal{O}_G is the structural sheaf associated with the superalgebra of its global sections $\mathcal{O}(G)$, so that the morphisms μ^*, i^*, e^* are actually defined as the appropriate sheaf morphisms, corresponding to μ, i, e , multiplication, inverse and unit in the algebraic supergroup $G = \underline{\text{Spec}}\mathcal{O}(G)$, corresponding to the SHCP (G_0, \mathfrak{g}) .

Proposition 3.11. *Let (G_0, \mathfrak{g}) be a SHCP, with G_0 an affine group scheme and let \mathcal{O}_G as in 3.1. Then $G := (G_0, \mathcal{O}_G)$ is a supergroup scheme.*

Proof. In proposition 3.10 we have seen that $\mathcal{O}(G) := \text{Hom}_{\mathcal{U}(\mathfrak{g}_0)}(\mathcal{U}(\mathfrak{g}), \mathcal{O}_{G_0}(G_0))$ has an Hopf superalgebra structure, moreover by 3.4 it is globally split. Hence we only need to prove that $G = \underline{\text{Spec}}\mathcal{O}(G)$. Clearly the topological spaces underlying the superspaces $G = (\overline{G_0}, \overline{\mathcal{O}_G})$ and $\underline{\text{Spec}}\mathcal{O}(G)$ are homeomorphic. We only need to show that $\mathcal{O}_{\mathcal{O}(G)} \cong \mathcal{O}_G$, where $\mathcal{O}_{\mathcal{O}(G)}$ denotes the structural sheaf associated with the superring $\mathcal{O}(G)$. We set up a morphism:

$$\begin{array}{ccc} \phi & \mathcal{O}_G(U) & \longrightarrow \mathcal{O}_{\mathcal{O}(G)}(U) \\ s : \mathcal{U}(\mathfrak{g}) \longrightarrow \mathcal{O}_{G_0}(U) & \mapsto & \phi(s) : U \longrightarrow \coprod_{x \in U} \mathcal{O}(G)_x \end{array}$$

where $\phi(s)$ is defined as follows. Any $s \in \mathcal{O}_G(U)$ gives raise naturally to $s_x : \mathcal{U}(\mathfrak{g}) \longrightarrow \mathcal{O}_{G_0}(U) \longrightarrow \mathcal{O}_{G_0,x}$. Since as a $\mathcal{U}(\mathfrak{g}_0)$ module, $\mathcal{U}(\mathfrak{g})$ is finitely generated, say by N generators, once we fix those generators, s_x is equivalent to the choice of N elements in $\mathcal{O}_{G_0,x}$. Since likewise $\mathcal{O}(G)_x$ is finitely generated by N elements as free $\mathcal{O}_{G_0,x}$ -module (those N elements corresponds dually to the generators of $\mathcal{U}(\mathfrak{g})$ as $\mathcal{U}(\mathfrak{g}_0)$ -module), we have that s_x can be viewed as an element of $\mathcal{O}(G)_x$. So we define:

$$\phi(s)(x) = s_x, \quad x \in U.$$

We leave to the reader the check that ϕ is a sheaf isomorphism. \square

Theorem 3.12. *The category of algebraic SHCP is equivalent to the category of affine algebraic supergroups.*

Proof. We need to establish a one to one correspondence between the objects and the morphisms.

As for the objects, if (G_0, \mathfrak{g}) is an algebraic SHCP, we can define an affine algebraic supergroup defining the following Hopf superalgebra (see 3.10):

$$\mathcal{O}(G_0, \mathfrak{g}) = \underline{\text{Hom}}_{\mathcal{U}(\mathfrak{g}_0)}(\mathcal{U}(\mathfrak{g}), \mathcal{O}(G_0)).$$

Vice-versa, if we have an algebraic supergroup, we can find right away the SHCP associated to it. What we need to show is that these operations are one the inverse of the other, that is:

$$\mathcal{O}(G_0, \mathfrak{g}) \cong \mathcal{O}(G)$$

where G_0 is the algebraic group underlying G and $\mathfrak{g} = \text{Lie}(G)$. Certainly they are isomorphic as $\mathcal{O}(G_0)$ -modules, since they have the same reduced part and, by a result of Masuoka [16], they both can be written as $\mathcal{O}(G_0) \otimes \wedge$ for some exterior algebra \wedge , but being their odd dimension the same, the two exterior algebras are isomorphic.

We can set a map:

$$\begin{array}{ccc} \eta^* : \mathcal{O}(G) & \longrightarrow & \mathcal{O}(G_0, \mathfrak{g}) \\ s & \mapsto & \bar{s} : X \mapsto (-1)^{|X|} |D_X(s)| \end{array}$$

where $D_X(s) = (1 \otimes X)\mu^*$. This is a well defined morphism of Hopf superalgebras and $X \mapsto (-1)^{|X|} |D_X(s)|$ is a $\mathcal{U}(\mathfrak{g}_0)$ -morphism. This is done precisely in the same way as in the proof of 3.8.

We now want to show that η^* is surjective. This will imply that η^* is an isomorphism. In fact the two given supergroups $G = \underline{\text{Spec}} \mathcal{O}(G)$ and $\overline{G} = \underline{\text{Spec}} \mathcal{O}(G_0, \mathfrak{g})$ are smooth superschemes, with the same underlying topological space and same Lie superalgebra (hence the same superdimension), and η^* induces an injective morphism $\eta : \overline{G} \longrightarrow G$ (see [9] sec. 2).

For the surjectivity of η^* , we need to show that for each morphism of $\mathcal{U}(\mathfrak{g}_0)$ -modules $\bar{s} : \mathcal{U}(\mathfrak{g}) \longrightarrow \mathcal{O}(G_0)$, there exists $s \in \mathcal{O}(G)$, such that $\bar{s}(X) = (-1)^{|X|} |D_X(s)|$. Since $\mathcal{U}(\mathfrak{g}) \cong \mathcal{U}(\mathfrak{g}_0) \otimes \wedge(\mathfrak{g}_1)$ (see theorem 3.4) and \bar{s} is a morphism of $\mathcal{U}(\mathfrak{g}_0)$ -modules, \bar{s} is determined by $\bar{s}(\gamma(X^I))$ for $X^I = X_1^{i_1} \dots X_n^{i_n}$, with X_i a basis for \mathfrak{g}_1 and $i_j = 0, 1$ (again refer to 3.4). Notice that $X_i = \gamma(X_i)$. Since X_1, \dots, X_n are linearly independent, also the corresponding left invariant vector fields D_{X_1}, \dots, D_{X_n} will be linearly

independent at each point. Let $D_{\gamma(X)}$ denote the left invariant differential operator corresponding to $\gamma(X) \in \mathcal{U}(\mathfrak{g})$. Notice that fixing a suitable basis in $\mathcal{U}(\mathfrak{g})$, the linear morphism $X \mapsto \gamma(X)$ corresponds to an upper triangular matrix and sends linearly independent vectors to linearly independent vectors. Consider the equation $(-1)^{|X^I|} |D_{\gamma(X^I)} s| = \bar{s}(X^I)$, for $X^I = X_1^{i_1} \dots X_n^{i_n}$ a monomial in $\wedge(\mathfrak{g}_1)$. This is an equation where each D_{X_i} appearing in the expression for $D_{\gamma(X^I)}$ can be expressed as

$$D_{X_i} = \sum a_i \partial_{x_{ij}}, \quad p(a_i) \neq p(x_{ij})$$

where x_{ij} are global coordinates on $\mathrm{GL}_{m|n} \supset G$ (regardless of their parity).

Since $D_{X_1}^{i_1} \dots D_{X_n}^{i_n}$ are linearly independent by the PBW theorem (see also proposition 2.11), also $D_{\gamma(X)}$ will be linearly independent and $(-1)^{|X|} |D_{\gamma(X^I)}| = \bar{s}(X^I)$ will yield a solution

$$\partial_{x_{i_1 j_1}} \dots \partial_{x_{i_r j_r}} s = a_{i_1 j_1 \dots i_r j_r}$$

for all $i_1 j_1 \dots i_r j_r$ so that

$$s = \sum a_{i_1 j_1 \dots i_r j_r} x_{i_1 j_1} \dots x_{i_r j_r}.$$

We leave to the reader the correspondence between morphisms. \square

Example 3.13. We want to verify explicitly the surjectivity of η^* in the case of $\mathrm{GL}(1|1)$ and make few remarks on how to extend the calculation to the case of $G = \mathrm{GL}(m|n)$. Let $\mathcal{O}(\mathrm{GL}(1|1)) = k[a_{11}, a_{22}, \alpha_{12}, \alpha_{21}][a_{11}^{-1}, a_{22}^{-1}]$. Let D_{12} and D_{21} denote the left invariant vector fields corresponding to the generators $\partial_{\alpha_{12}}, \partial_{\alpha_{21}}$ of $\mathrm{Lie}(G)_1$:

$$D_{12} = (1 \otimes \partial_{\alpha_{12}}) \mu^* = a_{11} \partial_{\alpha_{12}} + \alpha_{21} \partial_{a_{22}}$$

$$D_{21} = (1 \otimes \partial_{\alpha_{21}}) \mu^* = \alpha_{12} \partial_{a_{11}} + a_{22} \partial_{\alpha_{21}}$$

$$\gamma(D_{12} D_{21}) = 1/2(D_{12} D_{21} - D_{21} D_{12}) = 1/2(a_{11} \partial_{a_{11}} - a_{22} \partial_{a_{22}}) +$$

$$+ a_{11} a_{22} \partial_{\alpha_{12}} \partial_{\alpha_{21}} + \text{terms with coefficients in } J_{\mathcal{O}(\mathrm{GL}(1|1))}$$

where $J_{\mathcal{O}(\mathrm{GL}(1|1))}$ denotes as usual the ideal generated by the odd elements. Notice that the terms with coefficients in $J_{\mathcal{O}(\mathrm{GL}(1|1))}$ do not contribute in the

expression $|D_{\gamma(D_{12}D_{21})}s|$. For the same reason, notice that the term $a_{11}\partial_{a_{11}} - a_{22}\partial_{a_{22}}$ will give a contribute only if applied to s^0 , and consequently can be considered not as unknown, but as a known term. This is important in case one wants to generalize this procedure to $\text{GL}(m|n)$; in fact only the terms containing only odd derivations will produce new quantities to be determined.

Given $\bar{s} : \mathcal{U}(\mathfrak{g}) \longrightarrow \mathcal{O}(G_0)$ we want to determine $s \in \mathcal{O}(G)$, with $\eta^*(s) = \bar{s}$. Since $\text{Lie}(\text{GL}(1|1)_1) = \langle \partial_{\alpha_{12}}, \partial_{\alpha_{21}} \rangle$, \bar{s} is determined once we know its image on $\wedge \text{Lie}(\text{GL}(1|1)_1)$ that is

$$s^0 = \bar{s}(1), \quad s^{12} = \bar{s}(\partial_{\alpha_{12}}), \quad s^{21} = \bar{s}(\partial_{\alpha_{21}}), \quad s^{12,21} = \bar{s}(\gamma(\partial_{\alpha_{12}}\partial_{\alpha_{21}})).$$

Consequently the s we want to determine must satisfy the equations:

$$\begin{aligned} s^0 &= |1s| \\ s^{12} &= -|a_{11}\partial_{\alpha_{12}}s + \alpha_{21}\partial_{a_{22}}s| \\ s^{21} &= -|\alpha_{12}\partial_{a_{11}}s + a_{22}\partial_{\alpha_{21}}s| \\ s^{12,21} &= |1/2(a_{11}\partial_{a_{11}}s - a_{22}\partial_{a_{22}}s) + a_{11}a_{22}\partial_{\alpha_{12}}\partial_{\alpha_{21}}s| \end{aligned}$$

A simple calculation gives us:

$$\begin{aligned} s &= s^0 + \frac{\alpha_{12}s^{12}}{a_{11}} - \frac{\alpha_{21}s^{21}}{a_{22}} + \\ &+ \left[s^{12,21} - \frac{1}{2}(a_{11}\partial_{a_{11}}s^0 - a_{22}\partial_{a_{22}}s^0) \right] \frac{\alpha_{12}\alpha_{21}}{a_{11}a_{22}}. \end{aligned}$$

There is no conceptual obstacle to extend this calculation to the case of $G = \text{GL}(m|n)$. If $\mathcal{O}(G) = k[a_{ij}, \alpha_{kl}][d_1^{-1}, d_2^{-1}]$ where $d_1 = \det(a_{ij})_{\{1 \leq i, j \leq m\}}$ and $d_2 = \det(a_{ij})_{\{m+1 \leq i, j \leq m+n\}}$, we have that the left invariant vector fields are given by:

$$X_{ij} = (1 \otimes \partial_{x_{ij}}) \mu^* = \sum_k x_{ki} \partial_{x_{kj}}$$

where x_{ij} denote the coordinates on $\text{GL}(m|n)$ regardless of their parity. We can then repeat the calculation we did above. Notice that any even derivation appearing in the expression $|D_{\gamma(X)}s|$ will affect only $s^0 = |1s|$ since we are taking the reduction modulo the ideal of the odd nilpotents.

In the following remark we clarify the relation between the Hopf superalgebra $\mathcal{O}(G) = \text{Hom}(\mathcal{U}(\mathfrak{g}), \mathcal{O}(G_0))$ associated to the SHCP (G_0, \mathfrak{g}) and the distribution superalgebra $D(G)$ of the supergroup G (also naturally associated to the same SHCP).

Remark 3.14. For an affine supergroup G , the superalgebra of distributions $D(G)$ has a natural Hopf superalgebra structure as we detail in 2.15. Such structure is inherited by $k|G| \otimes \mathcal{U}(\mathfrak{g})$ through the linear isomorphism with $D(G)$ detailed in 2.16. The superalgebra of global sections of G , $\mathcal{O}(G) = \text{Hom}(\mathcal{U}(\mathfrak{g}), \mathcal{O}(G_0))$ can then be naturally viewed as a subspace of $D(G)^* \cong (k|G| \otimes \mathcal{U}(\mathfrak{g}))^*$, since elements in $\mathcal{O}(G)$ arise as suitable morphisms $|G| \times \mathcal{U}(\mathfrak{g}) \rightarrow k$. One can then immediately verify that the Hopf superalgebra structure on $\mathcal{O}(G) \subset D(G)^*$ is precisely obtained by duality, from the Hopf superalgebra on $D(G)$ suitably restricting the comultiplication, counit and antipode morphisms.

4 Action of supergroups and SHCP's

In this section we want to relate the action of an analytic of algebraic supergroup G on a supermanifold or superscheme M , with the action of the corresponding SHCP (G_0, \mathfrak{g}) on M . We first introduce a (well known) definition.

Definition 4.1. A morphism

$$a: G \times M \longrightarrow M$$

is called an *action* of G on M if it satisfies

$$a \circ (\mu \times \mathbb{1}_M) = a \circ (\mathbb{1}_G \times a) \tag{8a}$$

$$a \circ \langle \hat{e}, \mathbb{1}_M \rangle = \mathbb{1}_M \tag{8b}$$

In the functor of points notation, this is the same as:

1. $1 \cdot x = x, \forall x \in M(T)$, 1 the unit in $G(T)$,
2. $(g_1 g_2) \cdot x = g_1 \cdot (g_2 \cdot x), \forall x \in M(T), \forall g_1, g_2 \in G(T)$.

where T is a supermanifold (resp. a superscheme) and $M(T) = \text{Hom}(T, M)$ are the T -points of M .

If an action a of G on M is given, then we say that G *acts* on M .

If (G_0, \mathfrak{g}) is an analytic SHCP, we can define what it means for (G_0, \mathfrak{g}) to act on a supermanifold.

Definition 4.2. We say that the SHCP (G_0, \mathfrak{g}) acts on a supermanifold M , if there are

1. an action

$$\underline{a}: G_0 \times M \longrightarrow M \quad (9)$$

$\underline{a}: a \circ (j|_{G|} \longrightarrow G \times \mathbb{1}_M)$ of the reduced Lie group G_0 on the supermanifold M ;

2. a representation

$$\begin{aligned} \rho_a: \mathfrak{g} &\longrightarrow \text{Vec}(M)^{\text{op}} \\ X &\mapsto (X \otimes \mathbb{1}_{\mathcal{O}(M)}) a^* \end{aligned} \quad (10)$$

of the super Lie algebra \mathfrak{g} of G on the opposite of the Lie superalgebra of vector fields over M .

and the two morphisms satisfy the following compatibility relations

$$\rho_a|_{\mathfrak{g}_0}(X) = (X \otimes \mathbb{1}_{\mathcal{O}(M)}) \underline{a}^* \quad \forall X \in \mathfrak{g}_0 \quad (11a)$$

$$\rho_a(g.Y) = (\underline{a}^{g^{-1}})^* \rho_a(Y) (\underline{a}^g)^* \quad \forall g \in |G|, Y \in \mathfrak{g} \quad (11b)$$

where $a^g: M \rightarrow M$, $a^g := a \circ \langle \hat{g}, \mathbb{1}_M \rangle$

The next proposition tells us that actions of a SHCP correspond bijectively to actions of the corresponding analytic supergroup.

Proposition 4.3. *Let G be an analytic supergroup acting on a supermanifold M . Then there is an action of the SHCP $(G_0, \text{Lie}(G))$ on M . Conversely, given an action of the SHCP (G_0, \mathfrak{g}) on M , there is a unique action $a_\rho: G \times M \longrightarrow M$ of the analytic supergroup G corresponding to the given SHCP on M whose reduced and infinitesimal actions are the given ones. If U is an open subset of M , we have*

$$\begin{aligned} a_\rho^*: \mathcal{O}_M(U) &\longrightarrow \text{Hom}_{\mathcal{U}(\mathfrak{g}_0)}(\mathcal{U}(\mathfrak{g}), (C_{G_0}^\infty \hat{\otimes} \mathcal{O}_M)(|a|^{-1}(U))) \\ f &\mapsto \left[X \mapsto (-1)^{|X|} (\mathbb{1}_{C^\infty(G_0)} \otimes \rho(X)) \underline{a}^*(f) \right] \end{aligned} \quad (12)$$

Proof. Let us check that $a_\rho^*(f)$ is $\mathcal{U}(\mathfrak{g}_0)$ -linear. For all $X \in \mathcal{U}(\mathfrak{g})$ and $Z \in \mathfrak{g}_0$ we have

$$\begin{aligned} a_\rho^*(f)(ZX) &= (-1)^{|X|} (\mathbb{1} \otimes \rho(ZX)) \underline{a}^*(f) \\ &= (-1)^{|X|} (\mathbb{1} \otimes \rho(X)) (\mathbb{1} \otimes Z_e \otimes \mathbb{1}) (\mathbb{1} \otimes \underline{a}^*) \underline{a}^*(f) \\ &= (-1)^{|X|} (\mathbb{1} \otimes \rho(X)) (\mathbb{1} \otimes Z_e \otimes \mathbb{1}) (\tilde{\mu}^* \otimes \mathbb{1}) \underline{a}^*(f) \\ &= (\tilde{D}_Z \otimes \mathbb{1}) [a_\rho^*(f)(X)] \end{aligned}$$

We now check that a_ρ^* is a superalgebra morphism.

$$\begin{aligned}
[a_\rho^*(f_1) \cdot a_\rho^*(f_2)](X) &= m_{C^\infty(G_0) \hat{\otimes} \mathcal{O}(M)}[a^*(f_1) \otimes a^*(f_2)]\Delta(X) \\
&= (-1)^{|X|} m\left[(\mathbb{1} \otimes \rho(X_{(1)}))\underline{a}^*(f_1) \otimes (\mathbb{1} \otimes \rho(X_{(2)}))\underline{a}^*(f_2)\right] \\
&= (-1)^{|X|} (\mathbb{1} \otimes \rho(X))(\underline{a}^*(f_1) \cdot \underline{a}^*(f_2)) \\
&= a_\rho^*(f_1 \cdot f_2)(X)
\end{aligned}$$

where $f_i \in \mathcal{O}(M)$ and $X_{(1)} \otimes X_{(2)}$ denotes $\Delta(X)$. Concerning the “associative” property, we have that, for $X, Y \in \mathcal{U}(\mathfrak{g})$ and $g, h \in G_0$,

$$\begin{aligned}
[(\mu^* \otimes \mathbb{1})a_\rho^*(f)](X, Y)(g, h) &= [a_\rho^*(f)](h^{-1}.XY)(gh) \\
&= (-1)^{|X|+|Y|+|X||Y|} \rho(Y) \rho(h^{-1}.X) (\underline{a}^{gh})^*(f) \\
&= (-1)^{|X|+|Y|+|X||Y|} \rho(Y) (\underline{a}^h)^* \rho(X) (\underline{a}^g)^*(f) \\
&= [(\mathbb{1} \otimes a_\rho^*)a_\rho^*(f)](X, Y)(g, h)
\end{aligned}$$

and, finally, $(\text{ev}_e \otimes \mathbb{1})a_\rho^*(f) = \rho(1) = f$.

Uniqueness can be proved as follows. Let a be an action of G on M and let (\underline{a}, ρ_a) be as in prop. 4.3. If $f \in \mathcal{O}_M(U)$, then

$$\begin{aligned}
a^*(f) &\in (\text{Hom}_{\mathcal{U}(\mathfrak{g}_0)}(\mathcal{U}(\mathfrak{g}), C_{G_0}^\infty) \hat{\otimes} \mathcal{O}_M)(|a|^{-1}(U)) \cong \\
&\cong \text{Hom}_{\mathcal{U}(\mathfrak{g}_0)}(\mathcal{U}(\mathfrak{g}), (C_{G_0}^\infty \hat{\otimes} \mathcal{O}_M)(|a|^{-1}(U)))
\end{aligned}$$

hence, using eq. (8a) and the fact that ρ_a is an antihomomorphism, for all $X \in \mathcal{U}(\mathfrak{g})$

$$\begin{aligned}
a^*(f)(X) &= (-1)^{|X|} [(D_X \otimes \mathbb{1})a^*(\phi)](1) \\
&= (-1)^{|X|} (\mathbb{1} \otimes \rho_a(X))(a^*(f)(1)) \\
&= (-1)^{|X|} (\mathbb{1} \otimes \rho_a(X))\underline{a}^*(f)
\end{aligned}$$

□

Let us now assume G is an affine algebraic supergroup over a field of characteristic zero and (G_0, \mathfrak{g}) is the corresponding SHCP and furthermore assume they are acting on a supervariety M , the definition 4.2 being the same, taking the morphisms in the appropriate category.

We state the analogue of the proposition 4.3 in the algebraic setting, its proof being essentially the same.

Proposition 4.4. *Let G be an algebraic supergroup acting on a supervariety M (not necessarily affine). Then there is an action of the SHCP $(G_0, \text{Lie}(G))$ on M . Conversely, given an algebraic action of the algebraic SHCP (G_0, \mathfrak{g}) on M , there is a unique action $a_\rho: G \times M \rightarrow M$ of the algebraic supergroup G corresponding to the given SHCP on M whose reduced and infinitesimal actions are the given ones. If U is an open subset of M , we have*

$$\begin{aligned} a_\rho^*: \mathcal{O}_M(U) &\longrightarrow \text{Hom}_{\mathcal{U}(\mathfrak{g}_0)}(\mathcal{U}(\mathfrak{g}), (\mathcal{O}_{G_0} \otimes \mathcal{O}_M)(|a|^{-1}(U))) \\ f &\mapsto \left[X \mapsto (-1)^{|X|} (\mathbb{1}_{\mathcal{O}(G_0)} \otimes \rho(X)) \underline{a}^*(f) \right] \end{aligned} \quad (13)$$

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